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VASSILIEV INVARIANTS OF TWO COMPONENT LINKS AND THE CASSON–WALKER INVARIANT

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In this paper we will use the Casson–Walker invariant of three-manifolds to define a Vassiliev invariant of two-component links. This invariant, λ, is shown to have a simple combinatorial definition, making it easy to compute. In addition to placing λ in the general theory of Vassiliev invariants, we will consider applications. For example, it is seen that λ offers an easily computed obstruction to a link being amphicheiral and we give a quick proof that amphicheiral links cannot have linking number equal to 2 modulo 4.

Let L be a two-component link in S^3 with linking number n and let M_L denote the three-manifold that results from 0-surgery on L. For $n \neq 0$, define the rational invariant $\mu(L)$ to be the Casson–Walker invariant of M_L [1, 16]; for $n = 0$, let $\mu(L)$ be the third coefficient of the Conway polynomial of L. Finally, so that we can work over the integers, define λ by setting

$$\lambda(L) = \begin{cases} \mu(L) & \text{if } n = 0 \\ \frac{n^2}{2}\mu(L) + \frac{n^3 - n}{12} & \text{if } n \neq 0. \end{cases}$$

We prove the following.

THEOREM A. (1) *$\lambda(L)$ defines a Type* 1 *integral Vassiliev invariant on the space $\mathscr{S}_n$ of singular links of two disjoint components with linking number n.*

(2) *The group of integral Type* 1 *invariants on $\mathscr{S}_n$ modulo the subgroup of Type* 0 *invariants is rank one, generated by λ if n is even, and by $\frac{1}{2}\lambda$ if n is odd.*

Note the emphasis on disjoint; each component may have self-intersections, but the components do not intersect. The distinction between this setting for Vassiliev invariants and the more usual one in which the components are not necessarily disjoint is critical. For instance, in the standard setting the finiteness of the rank of Type r invariants is a simple result. Here finiteness is not at all obvious and we conjecture that, in fact, for Type r, with $r > 1$, the rank is infinite.

If we were considering the simpler case of rational instead of integral invariants, the theorem would remain true with μ replacing λ. In addition to showing that λ is integral, we will prove that λ is congruent modulo n to the third coefficient of the Conway polynomial.

Our approach also yields a simple combinatorial understanding of λ, including a quick computational method which we now describe. Let L_+ and L_- be two linking number

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n links which differ by a single crossing change, and denote by L_s the intermediate singular link obtained by replacing the crossing with a double point. The two loops obtained by starting at the double point of L_s and following the component of the link until one returns to the double point will be called the *lobes* of the singular link L_s.

Denote by κ the linking number of one of the lobes of L_s with the other link component; the other lobe has linking number $n - \kappa$. (This notation will be used throughout the paper.) The product $\kappa(n - \kappa)$ is independent of which lobe one takes first, and forms a fundamental quantity associated to a singular link with one singular point.

The following two theorems imply the first part of Theorem A and are of independent interest.

THEOREM B. *If χ is an integral link invariant that satisfies the crossing change formula*

$$\chi(L_+) - \chi(L_-) = \kappa(n - \kappa)$$

where $\kappa(n - \kappa)$ is associated to the intermediate singular link as above, then χ is Type 1 on $\mathscr{S}_n$ and is onto $\mathbf{Z}$ if n is even, and onto $2\mathbf{Z}$ or $2\mathbf{Z} + 1$ if n is odd.

THEOREM C. *The invariant λ satisfies the crossing change formula of Theorem B.*

Using the definition alone it is difficult to compute λ, since μ is difficult to compute. This theorem shows, however, that computing the *difference* of the value of λ on two links is simple. In particular, knowing the value of λ of just one link for each linking number makes its computation easy. We provide a family of links, the suitably oriented $(2, n)$-torus links, for which λ is trivial.

As mentioned above, the standard approach to Vassiliev invariants of knots and links considers the space of all singular links, $\bar{\mathscr{S}}$, rather than the space of links with disjoint components, $\mathscr{S}$, that we have been considering. We are able to relate the two approaches with the following result.

THEOREM D. (1) *No Type 2 invariant on $\bar{\mathscr{S}}$ restricts to be nontrivial Type 1 on $\mathscr{S}$. In particular, λ does not have a Type 2 lifting to $\bar{\mathscr{S}}$.*

(2) *The extension of λ to $\bar{\mathscr{S}}$ is a Type 3 invariant.*

Using Theorem A we derive the following theorem which gives an easy combinatorial method to compute the Casson–Walker invariant of 0-surgery on a two component link.

THEOREM E. *Let L be a link with linking number $n \neq 0$ and let M_L denote the manifold obtained by 0-surgery on L. Let L' denote the link obtained from L by reversing the orientation of one component of the mirror image of L. (So L' also has linking number n.) Then*

$$\mu(M_L) = (\lambda(L) - \lambda(L'))/n^2.$$

A link is called *amphicheiral* if it is isotopic to its mirror image—here we can ignore orientations and the ordering of the components. Notice that since λ is integral, the next theorem implies that an amphicheiral link cannot have linking number equal to 2 modulo 4.

THEOREM F. *If a two component link L with linking number n is amphicheiral, then $\lambda(L) = \frac{1}{12}(n^3 - n)$.*

This paper is organized as follows. In Section 1 we present a brief review of the basic theory of Vassiliev invariants in the context of examining Type 0 invariants on $\mathscr{S}_n$. Section 2 gives a proof of Theorem B. At this point we also present an example that will play an important role in Section 3 and Section 5; here it is used to illustrate how the crossing change formula yields a simple computational method. In Section 3 we prove Theorem C, and hence part (1) of Theorem A. The proof of Theorem A is completed in Section 4, where our most detailed argument shows that any Type 1 invariant must satisfy the crossing change formula given in Theorem B. The referee has noted that the construction of this section offers insight into the algebraic topology of the space $\mathscr{S}$. In fact, letting $\mathscr{L}_n$ denote the space of embedded links of linking number n, we show that $H^1(\mathscr{S}_n, \mathscr{L}_n; \mathbf{Z})$ is infinitely generated. Section 5 is devoted to the proof of Theorem D. The final section presents an application to amphicheirality (Theorem F), a discussion of periodicity, the proof of Theorem E, and the proof that λ reduces modulo n to the third coefficient of the Conway polynomial.

There are a number of good references for the basic theory of Vassiliev invariants. Either [2] or [4] would provide needed background material. Other important references include [5, 11, 25]. An extensive bibliography of the subject is contained in [3]. The work of Lin [12] on link homotopy invariants and Vassiliev invariants provides a useful contrast to the results of this paper. It is clear that any link invariant that restricts to be Type 0 on the space of singular links with disjoint components is a link homotopy invariant; Lin shows that basically all such are given by Milnor invariants.

One interesting problem that remains is to relate $\lambda(L)$ to other link invariants. For instance, for linking number 0, λ is the Sato–Levine invariant [6, 14]. Our initial investigations indicate that in general λ is not determined by such invariants as the HOMFLY or Kauffman polynomial, nor is it related to classical signatures of L. We expect, however, that alternative interpretations do exist. These issues are not pursued in this paper. Our goal here is first to initiate the study of Vassiliev invariants in the focused setting of links of a fixed number of components and to identify the unique role of λ in this study. Secondly, we are concerned with the applicability of the abstract results, demonstrated by the ease of calculation of λ, its use in computing Casson–Walker invariants, and its serving as an obstruction to amphicheirality and periodicity.

1. NOTATION AND TYPE 0 INVARIANTS

Let $\mathscr{S}$ denote the space of smooth oriented singular links of two disjoint components in S^3, where singularities consist only of transversal double points. (A double point is transversal if the tangent vectors at the double point are linearly independent.) Since linking numbers are well defined on $\mathscr{S}$, $\mathscr{S}$ has components $\mathscr{S}_n$ consisting of singular links of linking number n. We denote by $\mathscr{L}_n$ the subspace of $\mathscr{S}_n$ consisting of nonsingular links.

Suppose that χ is a link invariant defined on $\mathscr{L}_n$ taking values in an abelian group. Then χ defines an invariant on $\mathscr{S}_n$ inductively via the formula

$$\chi(L_s) = \chi(L_+) - \chi(L_-).$$

One says that χ is of *finite type*, or *Vassiliev*, if there is an integer r such that χ vanishes on all links with $r + 1$ or more double points. In this case χ is said to be of *Type r*. Notice that the set of invariants of Type r forms an abelian group and the groups are naturally nested; any Type r invariant is also of Type $r + 1$. We conclude this introduction with the proof of the following simple basic result concerning Type 0 invariants.

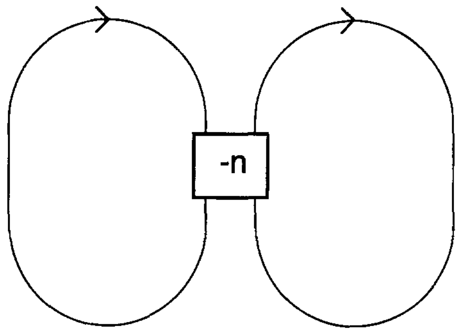

Fig. 1.

1.1. THEOREM. *The group of Type* 0 *integral invariants on* $\mathcal{S}_n$ *is rank one, isomorphic to* $\mathbf{Z}$.

Proof. Suppose that χ is a Type 0 invariant. Let L be a link in $\mathcal{L}_n$. There is a path α in $\mathcal{S}_n$ from L to the link denoted by $T(n)$ in Fig. 1. (In the illustration, the number in the box represents the number of full right-handed twists between the strands.) The value of χ is constant on α since, for a Type 0 invariant, crossing changes do not alter its value. Hence, χ is completely determined by its value on $T(n)$. This shows the rank of the group of invariants is at most 1. However, the constant functions on $\mathcal{L}_n$ defines Type 0 invariants. ■

2. THE CROSSING CHANGE FORMULA FOR TYPE 1 INVARIANTS

Section 4 presents the most difficult proof of this paper, that any Type 1 invariant, χ, on $\mathcal{L}_n$ satisfies the crossing change formula

$$\chi(L_+) - \chi(L_-) = d\kappa(n - \kappa), \qquad (*)$$

for some constant d, where L_+, L_- and κ are as defined in the introduction. Here we note the easy converse.

2.1. THEOREM. *If an invariant* χ *on* $\mathcal{L}_n$ *satisfies* $(*)$ *for some constant* d, *then* χ *is Type* 1.

Proof. The proof is trivial. If a link has two double points, by focusing on one of them we see that the value of χ is given as the difference of its value on two different links with one double point each. However, the value of $\kappa(n - \kappa)$ is the same for both of these, and hence this difference is 0. ■

A second simple result completes the proof of Theorem B.

2.2. THEOREM. *Suppose that* χ *is an integral invariant on* $\mathcal{L}_n$ *satisfying* $\chi(L_+) - \chi(L_-) = \kappa(n - \kappa)$ *where* L_+, L_- *and* κ *are as defined in the introduction. Then* χ *is onto the integers for n even, and onto* $2\mathbf{Z}$ *or* $2\mathbf{Z} + 1$ *for n odd.*

Proof. First note that if n is odd, then $k(n - k)$ is always even.

Given any link of linking number n, one can construct a crossing change for which the value of κ is arbitrary. Hence we need to show that the set of integers $\{k(n-k)\}_{k\in\mathbf{Z}}$ generates $\mathbf{Z}$ or $2\mathbf{Z}$, depending on whether n is even or odd respectively.

Consider the function $f(k) = k(n-k)$. A quick calculation shows that $f(k) - 2f(k+1) + f(k+2) = 2$. Hence, in the case of n odd, it is immediate that the values of $k(n-k)$ generates the even integers. If n is even, then $f(1)$ is odd, and since we have seen that 2 is also in the set of integers generated by $\{k(n-k)\}$, we are done. ■

Although it is in the next section that we demonstrate the existence of Type 1 invariants, at this point it may be useful to demonstrate the use of the crossing change formula for computational purposes. The example will be used in a central way in Sections 3 and 5.

Example. The link $T(n)$ illustrated in Fig. 1 can be modified to give a new linking number n link by reversing all its crossings and changing the orientation of one of the two components. This new link, $T'(n)$, along with $T(n)$ are illustrated in Fig. 2 in a way that facilitates our calculation, in this case with $n = 4$. Let χ be an invariant satisfying $\chi(L_+) - \chi(L_-) = \kappa(n-\kappa)$. Our result is that in this case

$$\chi(T(n)) - \chi(T'(n)) = -\tfrac{1}{6}(n^3 - n).$$

To see this, note that $T(n)$ differs from $T'(n)$ by $(n-1)$ crossing changes. Using the crossing change formula, we see that $\chi(T(n)) - \chi(T'(n)) = -\sum_{j=1}^{n-1} j\cdot(n-j) = -\frac{1}{6}(n^3 - n)$. (One simple consequence, once it is known that χ exists, is that $T(n)$ and $T'(n)$ are distinct for $n \neq 0, 1$ or -1. Of course, this can be shown by more elementary methods.)

3. CROSSING CHANGE FORMULA FOR THE CASSON–WALKER INVARIANTS

It is now our goal to show that the invariant λ defined in the introduction yields an integral invariant satisfying the crossing change formula $\lambda(L_+) - \lambda(L_-) = \kappa(n-\kappa)$. The case of $n = 0$ is an easy exercise concerning the Conway polynomial; it is a special case of Theorem 6.3 below. (It was first noticed by Jin [8], see also [9], that the Sato–Levine invariant satisfies $\chi(L_+) - \chi(L_-) = \kappa^2$; for linking number 0 links the Sato–Levine invariant equals the third coefficient of the Conway polynomial [6].)

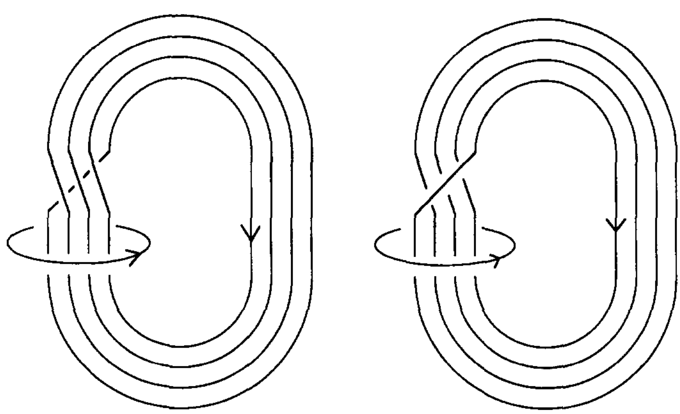

Fig. 2.

If $n \neq 0$, then we consider the three manifold M_L obtained by performing 0-surgery on both components of L. Then M_L is a rational homology sphere, with $H_1(M_L) = \mathbf{Z}/n \oplus \mathbf{Z}/n$. Walker [16] defined a rational invariant we call μ here of $\mathbf{Q}$-homology spheres, generalizing the Casson invariant of homology spheres [1]. A good reference for this material, including a discussion of linking numbers and Alexander polynomials in rational homology spheres is [16].

Suppose that L and L' are related by a single crossing change, from positive to negative. We need to compute the difference $\mu(M_L) - \mu(M_{L'})$. However, $M_{L'}$ is obtained from M_L by performing $+1$-surgery on a null homologous knot K in M_L, as illustrated schematically in Fig. 3(a). (See [13] for a discussion of such surgery diagrams.)

Since K is null homologous, it has an Alexander polynomial that is well defined up to rational multiples and multiplication by integer powers of t. Let $\Delta_K(t)$ denote this polynomial, normalized so that $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$. Walker [16, Theorem 4.2] gives a formula for the change of the value of μ under surgery of this sort, and in the present case the formula reduces to $\mu(M_L) - \mu(M_{L'}) = (d^2/dt^2)\,(\Delta_K)(1)$. Hence the crucial calculation is the following.

3.1. THEOREM. *The normalized Alexander polynomial for the knot K in the rational homology sphere M_L described above is*

$$\Delta_K(t) = -\frac{1}{n^2}\left[-\kappa(n-\kappa)t^{-1} - ((n-\kappa)^2 + \kappa^2) - \kappa(n-\kappa)t\right].$$

In particular, $\mu(M_L) - \mu(M_{L'}) = (2/n^2)\kappa(n-\kappa)$.

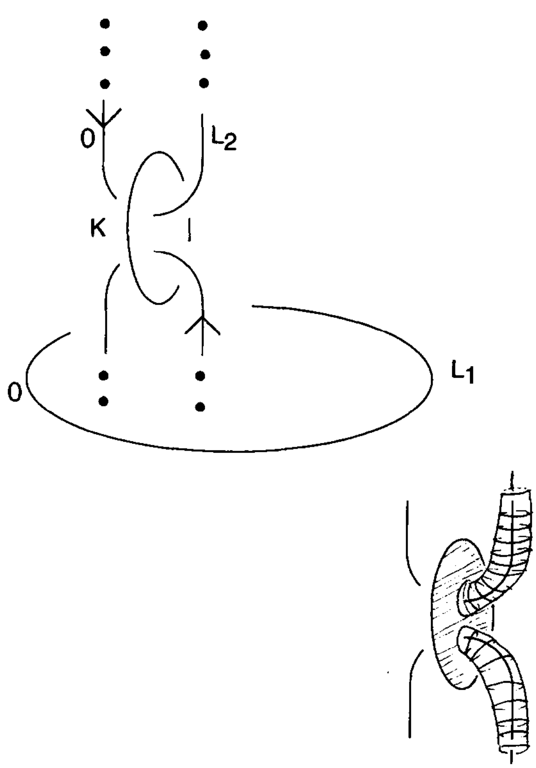

Fig. 3.

Proof. The proof uses the Seifert form of K to compute the polynomial. A Seifert surface F for K in M_L is illustrated in Fig. 3(b). As a basis $\{\alpha_1, \alpha_2\}$ for $H_1(F)$, let α_1 be a small linking circle to L_2 on F and let α_2 be a dual curve. (There is an indeterminacy in the choice of α_2; to be precise, we pick α_2 so that it links L_2 algebraically 0 times.) The Seifert form of K is given by the 2×2 matrix V, with entries $v_{ij} = lk(\alpha_i^+, \alpha_j)$, where α_i^+ is the positive push off and the linking number is computed in M_L.

If the Seifert matrix were computed using linking numbers in S^3 instead of M_L, then the resulting matrix would be

$$\begin{pmatrix} 0 & 1 \\ 0 & * \end{pmatrix}.$$

Lemma 3.3 below describes how to correct for the fact that we are working in M_L in terms of the surgery matrix of M_L and the linking numbers of α_1 and α_2 with the components of L. The result is

$$V = \begin{pmatrix} 0 & 1 - (\kappa/n) \\ -(\kappa/n) & * \end{pmatrix}.$$

The Alexander polynomial is given as $\det(V - tV^{\mathrm{T}})$; the theorem now follows. ∎

We can now complete the proof of Theorem A(1) Theorem C.

Proof of Theorem A(1) *and Theorem* C. What must be shown is that on $\mathscr{L}_n$ $(n \neq 0)$, $\lambda(L) = \frac{1}{2}n^2\mu(L) + \frac{1}{12}(n^3 - n)$ defines an integral Type 1 invariant which satisfies the crossing change formula

$$\lambda(L_+) - \lambda(L_-) = \kappa(n - \kappa).$$

It follows from Theorem 3.1 that λ satisfies the crossing change formula. Theorem B then implies that λ has Type 1.

It remains to show that this invariant is integral. We will show that λ vanishes on $T(n)$; this implies it is integral since the crossing change formula shows that the value of λ on any link differs from $\lambda(T(n))$ by an integer (an even integer if n is odd).

Notice that as unoriented links, $T'(n)$ is obtained from $T(n)$ by taking the mirror image. Thus as oriented manifolds, $M_{T(n)} = -M_{T'(n)}$ and so $\mu(T(n)) = -\mu(T'(n))$. Therefore:

$$\begin{aligned} n^2\mu(T(n)) &= \tfrac{1}{2}n^2(\mu(T(n)) - \mu(T'(n))) = \lambda(T(n)) - \lambda(T'(n)) \\ &= -\tfrac{1}{6}(n^3 - n). \end{aligned}$$

The last equality follows from the computation given at the end of Section 2. This immediately implies that $\lambda(T(n)) = 0$. ∎

We record for future use the following corollary to the proof.

3.2. Corollary. *The Casson–Walker invariant of* 0-*surgery on* $T(n)$ *equals* $(1 - n^2)/(6n)$. *In particular the invariant* λ *vanishes on the link* $T(n)$.

In order to compute the necessary linking numbers used in the proof of Theorem 3.1, we consider the following general situation. Let $\{J, K, L\}$ be a link in S^3 where L is an ordered multi-component link. Let M_L be a manifold obtained by surgery on L and let A be the surgery matrix, consisting of the linking numbers of L off the diagonal and the surgery coefficients on the diagonal. Let v_J *and* v_K be vectors expressing the homology classes of J and K in $S^3 - L$ in terms of the meridians of L. Finally, let $\mathrm{lk}(J, K)$ denote the linking number of J and K in S^3 and $\mathrm{lk}(J, K; M_L)$ denote the linking number of J and K, thought of as curves in M_L. This latter linking number is only defined if M_L is a rational homology sphere or, equivalently, if A is nonsingular over $\mathbf{Q}$. The linking number in M_L is a rational number.

3.3. LEMMA. *If M_L is a rational homology sphere, then* $\mathrm{lk}(J, K; M_L) = \mathrm{lk}(J, K) - v_J A^{-1} v_K^{\mathrm{T}}$.

Proof. (For the special case of integral homology spheres this lemma follows from results of [7].) We reduce the calculation of $\mathrm{lk}(J, K; M_L)$ to that of the linking numbers of the meridians to L in M_L as follows. Let F be a surface in $S^3 - L$ representing a homology from J to a collection of meridians of L. The intersection number of K with F is $\mathrm{lk}(J, K)$. Next, K is homologous to a collection of meridians of L, with the homology missing the previously selected meridians homologous to J. It remains to show that A^{-1} determines the linking number of meridians.

To compute the linking number of one meridian with another, the main step is to construct surfaces bounded by the meridians. Let δ_i be the disks attached to $S^3 - L$ in performing the surgery, and let Δ_i denote the surfaces in $S^3 - L$ formed by puncturing the surfaces bounded by the longitudes l_i of L in S^3. It is immediate that $\partial\delta_i = l_i + a_{ii}m_i$, where the m_i are the meridians of L. Moreover, $\partial\Delta_i = l_i - \sum_{j \neq i} a_{ij}m_j$, modulo surfaces contained in the boundary of $S^3 - L$. In vector notation, these equations can be combined and written as $\partial(\delta - \Delta) = Am$. Hence, rationally, $m = A^{-1}\partial(\delta - \Delta)$. The result now follows, noting that m_i misses all δ_i and intersects Δ_i in one point and misses all other Δ_j. ■

In the proof of Theorem 3.1 we take v_J to be $(1, 0)$ and v_K to be $(0, \kappa)$. The surgery matrix is

$$A = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}.$$

Example. It is now an easy exercise to generalize the example of $T(n)$ to compute the value of λ and the Casson–Walker invariant for a large class of links built from torus knots. Define $T(p, q)$ to be the link having one component the (p, q)-torus knot contained in a standard torus in S^3. The second component is the meridian to the torus, pushed off and oriented so that the linking number of the link is p. As a special case, $T(n) = T(n, -1)$.

Generalizing the calculation used for $T(n)$, one computes that

$$\lambda(T(p, q)) - \lambda(T(p, -q)) = \tfrac{1}{6} q \cdot (p^3 - p).$$

From its definition one has

$$\lambda(T(p, q)) - \lambda(T(p, -q)) = \tfrac{1}{2} p^2 \mu(M_{T(p, q)}) - \tfrac{1}{2} p^2 \mu(M_{T(p, -q)}).$$

However, $M_{T(p, q)} = -M_{T(p, -q)}$, so $\frac{1}{2} p^2 \mu(M_{T(p, q)}) = -\frac{1}{2} p^2 \mu(M_{T(p, -q)})$. Combining these calculations, one finds that

$$\mu(M_{T(p, q)}) = q \cdot (p^2 - 1)/6p \quad \text{and} \quad \lambda(T(p, q)) = \tfrac{1}{12}(q + 1) \cdot (p^3 - p).$$

(Compare this to Theorem E and its proof in Section 6.)

4. COMBINATORIAL PROPERTIES OF TYPE 1 INVARIANTS

The key to the proof of Theorem A (2) is the following theorem.

4.1. THEOREM. *If χ is a Type 1 invariant on $\mathscr{L}_n$ then it satisfies the crossing change formula*

$$\chi(L_+) - \chi(L_-) = d\,\kappa(n-\kappa)$$

for some d, where L_+, L_- and κ are as in the introduction.

To prove Theorem 4.1, we will first show that any Type 1 invariant χ is determined by its value on a particular set of links. This set consists of a doubly infinite family (rather than a singleton), but fortunately we can find a number of restrictive relationships between the values of χ on this family. These relations are described in Lemma 4.2 below. First we set up the notation.

Let χ be a Type 1 invariant on $\mathscr{S}_n$. By subtracting a Type 0 invariant, we can assume that $\chi(T(n)) = 0$. Since any link L in $\mathscr{L}_n$ can be turned into $T(n)$ by a series of crossing changes, the value of χ is clearly determined by its value on links with single double points. Furthermore, it is easy to show that any link with a single double point can be changed into one of the links, $S_1(k)$ and $S_2(k)$, illustrated in Fig. 4, again by changing crossings away from the double point. We denote the value of χ on $S_1(k)$ by χ_k^1 and its value on $S_2(k)$ by χ_k^2.

Since crossing changes create and then eliminate a second double point, and χ is Type 1, the value of χ is unchanged by such changes and is hence completely determined by the values χ_k^1 and χ_k^2. Relations between the links $S_i(k)$ provide the following relations between the values of χ.

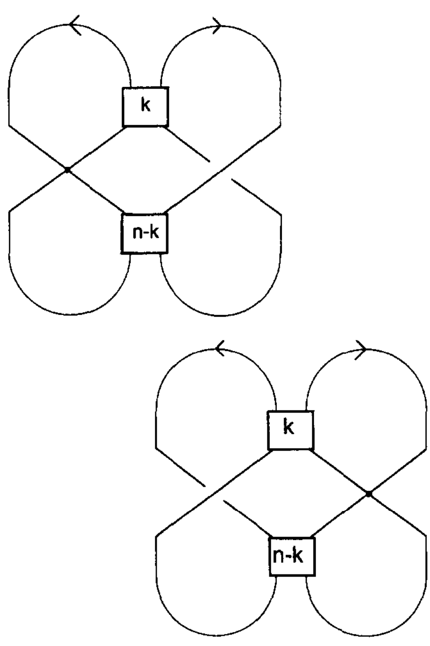

Fig. 4.

4.2. LEMMA. *In the notation above*:
(0) $\chi_0^i = 0$,
(1) $\chi_k^1 = \chi_k^2$,
(2) $\chi_k^1 = \chi_{n-k}^1$,
(3) *There is a constant* d *such that* $\chi_k^1 = d \cdot k(n-k)$.

Proof. (0) Notice that the links L_+ and L_- are isotopic, and hence the difference $\chi^i(L_+) - \chi^i(L_-) = 0$.

(1) The value of χ_k^1 is the difference of the value of χ on a pair of links. Similarly for χ_k^2. The two pairs of links that arise are easily seen to be isotopic.

(2) The links $S_1(k)$ and $S_1(n-k)$ are isotopic; consider a rotation of 180° about the horizontal axis. The proof of (2) follows.

(3) The proof of (3) is the most complicated. We begin by making the following essential observation. For any singular link L with one double point, the value of $\chi(L)$ can be determined as follows. Let α be one of the lobes, that is, one of the closed paths on L that begins at the double point and follows the corresponding component of L until it returns to the double point. The linking number of α with the other component is some integer, k. Clearly then, crossing changes in L will change it into either $S_1(k)$ or $S_2(k)$ without altering the value of χ. Hence $\chi(L) = \chi_k^1$. In the following we will denote this number by χ_k.

The next step in the proof of (3) is to develop the identity

$$\chi_k = \tfrac{1}{2}k(k-1)\chi_{-1} + \tfrac{1}{2}k(k+1)\chi_1. \qquad (*)$$

To prove this, consider the nonsingular link illustrated schematically in Fig. 5. (In the illustration we have $k = 3$. The thick curve on the right represents three parallel strands. We draw the strands separately only where the extra detail is necessary.) One can construct a new link by one of two procedures; either change the one self crossing of the left-hand component in the illustration, or change the k^2 crossing for the right-hand component. (These k^2 crossings occur in Fig. 5 where the thick band crosses over itself.) In both cases the resulting link greatly simplifies; in fact, both lead to the link $T'(n)$ connected sum with a torus knot. It follows easily from (0) that the connected sum does not alter the value of χ, so the two resulting links have the same value of χ. Hence, either the one crossing change or

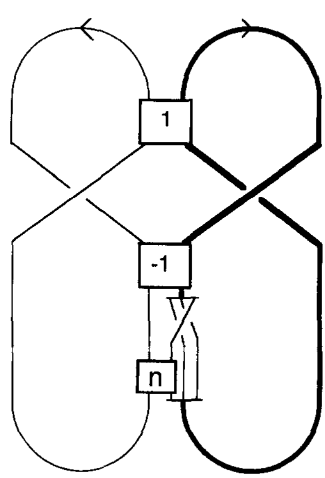

Fig. 5.

the k^2 changes lead to the same change in the value of χ. A careful counting argument, left to the reader, now gives (*). (Note that the left-hand side of (*) has one term while the right-hand side has k^2 terms.)

Using (*) along with $\chi_k = \chi_{n-k}$ we can attain two equations, first by setting $k = 0$ and then $k = 1$. On simplifying, we find that

$$(n(n-1)/2)\chi_{-1} + (n(n+1)/2)\chi_1 = 0 \tag{**}$$

and

$$(n-1)\chi_{-1} + (n+1)\chi_1 = 0. \tag{***}$$

Of course, over the integers (**) follows from (***), but for later computations in $\mathbf{Z}/r$ we need both. Now, using either equation, one can solve for either χ_{-1} or χ_1 in terms of the other over the rationals, and then substitute into (*). The desired result, (3), follows. ■

Proof (*Theorem A* (2)). We have now seen that a Type 1 invariant on $\mathscr{S}_n$ is determined by its value on $T(n)$ and the value of d in its crossing change formula. Hence, modulo Type 0 invariants, it is determined by the value of d. Hence, the rank is at most one, and since λ gives a nontrivial Type 1 invariant, the rank must be exactly one. Finally, by Theorem 2.2, λ (resp. $\frac{1}{2}\lambda$) is onto the integers for n even (resp. n odd), and hence must actually be the generator. ■

Our focus so far has been on integral invariants. Slight modifications give analogous results for invariants taking their value in finite groups. The following corollary gives a result analogous to Theorem A for Type 1 $\mathbf{Z}/r$-valued invariants.

4.3. Corollary. *For finite r, the group of Type 1 $\mathbf{Z}/r$-valued invariants modulo the group of Type 0 invariants is cyclic, isomorphic to $\mathbf{Z}/r$, generated by the $\mathbf{Z}/r$ reduction of λ for n even and $\frac{1}{2}\lambda$ for n odd.*

Proof. Using a direct sum decomposition, we can reduce to the case in which r is a prime power, say p^s. The calculations in the proof of Lemma 4.2 hold mod r, and so equations (**) and (***) continue to apply. There are two cases to consider.

In the first case, assume p is odd. Then one of $n - 1$ and $n + 1$ is prime to r, and hence is invertible in $\mathbf{Z}/r$. It follows, using (***) that one of χ_1 and χ_{-1} can be expressed in terms of the other, and we are as in the proof of Theorem 4.1.

In the case that p is even, if n is also even, then $n - 1$ is again invertible in $\mathbf{Z}/r$ and we are as above. If n is odd, we look instead to (**). Now, one of $\frac{1}{2}n(n-1)$ and $\frac{1}{2}n(n+1)$ is easily shown to be odd, and hence invertible in $\mathbf{Z}/r$, so we can again proceed as before. The rest of the argument is now identical to that of the case of integral invariants. ■

Remark. The referee has noted that the construction in the proof of Lemma 4.2 offers insight into the algebraic topology of the space $\mathscr{S}$.

Fix an integer n. We will show that this construction can be used to produce many nontrivial loops in $\pi_1(\mathscr{S}_n)$ which do not lie in the image of the map $\pi_1(\mathscr{L}_n) \to \pi_1(\mathscr{S}_n)$. (Recall that $\mathscr{S}_n$ denotes the space of two-component singular links with disjoint image and linking number n; $\mathscr{L}_n$ is the subspace of embedded links.) Choose some embedded link L with linking number n as a base point for $\mathscr{L}_n$ and $\mathscr{S}_n$.

First, note that $\pi_1(\mathscr{L}_n)$ is nontrivial. For example, a nontrivial loop can be constructed by viewing $T(n)$ as a knot in a solid torus and taking the isotopy of $T(n)$ to itself obtained by rotating the solid torus one revolution. Of course, $\mathscr{L}_n$ has many path components (one for each isotopy class of link of linking number n), and these path components may not all have the same fundamental group.

Let $A(n)$ denote the abelian group generated by elements χ_k with $k \in \mathbf{Z}$ subject to the relations $\chi_k = \chi_{n-k}$ and $\chi_0 = 0$. Thus $A(n)$ is free abelian, with generators the χ_k for $k \geqslant \frac{n}{2}$, $k \neq 0, n$.

We define a homomorphism

$$\Phi = (\Phi_1, \Phi_2) : \pi_1(\mathscr{S}_n) \to A(n) \oplus A(n)$$

by the following method. Given $\alpha \in \pi_1(\mathscr{S}_n, \{L\})$, view α as a homotopy from L to itself. By a small perturbation relative to the endpoints, one can assume that $\alpha(t)$ is an embedding for all $t \in [0, 1]$ except for $t = t_1 < t_2 < \cdots < t_k$; in this case $\alpha(t_i)$ is a self-transverse immersion. Define $\Phi_1(\alpha)$ to be the sum with signs over all the double points occuring on the first component of χ_κ, with κ as before. Similarly define Φ_2 using the second component.

The proof that $\Phi(\alpha)$ is well defined is easily understood by considering a generic homotopy between two loops. Whenever double points cancel, they cancel with sign and with the same pair $(\kappa, n - \kappa)$ with one exception, namely, when a singularity occurs in a homotopy, a "kink" appears and a single double-point with κ equal to 0 or n can be created or eliminated. For this reason one must ignore the double points corresponding to $\kappa = 0$ or n. (If we were to consider the space of *immersions* instead of $\mathscr{S}_n$, then one could count these double points as well.) A routine argument shows Φ to be a homomorphism. Notice that the composite

$$\pi_1(\mathscr{L}_n) \to \pi_1(\mathscr{S}_n) \xrightarrow{\Phi} A(n) \oplus A(n)$$

is the zero map, since isotopies have no double points.

Let Ψ: $A(n) \oplus A(n) \to \mathbf{Z}$ be the homomorphism defined by

$$\Psi(\chi_k, 0) = k(n - k) = \Psi(0, \chi_k).$$

Then the fact that the invariant λ is well defined is equivalent to the assertion that the composite

$$\pi_1(\mathscr{S}_n) \xrightarrow{\Phi} A(n) \oplus A(n) \xrightarrow{\Psi} \mathbf{Z}$$

is zero.

The loop in $\mathscr{S}_n$ constructed in the proof of part (3) of Lemma 4.2 is mapped by Φ to the element

$$\left(-\chi_k, \frac{k(k-1)}{2}\chi_{-1} + \frac{k(k+1)}{2}\chi_1\right)$$

by (*) in the proof of Lemma 4.2. This immediately implies that the map $\pi_1(\mathscr{L}_n) \to \pi_1(\mathscr{S}_n)$ is not surjective. In fact, a simple algebraic argument as in the proof of Lemma 4.2 shows that the image of Φ equals the kernel of Ψ. Hence Φ has an infinitely generated image, with cokernel isomorphic to $\mathbf{Z}$. In particular, $H_1(\mathscr{S}_n, \mathscr{L}_n; \mathbf{Z})$ maps onto an infinitely generated free abelian group and hence $H^1(\mathscr{S}_n, \mathscr{L}_n; \mathbf{Z})$ is infinitely generated.

One can ask whether the kernel of Φ equals the image of the map $\pi_1(\mathscr{L}_n) \to \pi_1(\mathscr{S}_n)$. A consequence of the main result of [10] is that at least for $n = 0$ there is a surjective homomorphism from the kernel of $\Phi: \pi_1(\mathscr{S}_0) \to A(0) \oplus A(0)$ to $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ which vanishes on the image of $\pi_1(\mathscr{L}_0)$.

5. LIFTING INVARIANTS TO NONDISJOINT LINKS

In this section we determine to what extent Type 1 invariants defined on the space $\mathscr{S}$ extend to finite type invariants on $\bar{\mathscr{S}}$, the space of nondisjoint 2 component singular links.

Let R be one of the rings $\mathbf{Z}$, $\mathbf{Q}$ or $\mathbf{Z}/r$. Given two functions

$$f, d: \mathbf{Z} \to R$$

we have constructed in the previous sections a Type 1 Vassiliev invariant $\chi_{d,f}$ on $\mathscr{S}$ defined via the two conditions

$$\chi_{d,f}(L_+) - \chi_{d,f}(L_-) = d(n)\cdot\kappa(n-\kappa) \quad \text{on } \mathscr{S}_n,$$

and

$$\chi_{d,f}(T(n)) = f(n).$$

Furthermore, these choices define a link invariant on $\bar{\mathscr{S}}$ via the crossing change formula.

The basic question can now be formulated: For what choices of d and f does $\chi_{d,f}$ define a finite type invariant on $\bar{\mathscr{S}}$, and what is its type, in terms of d and f? Of course, we know that if there are two or more crossings each involving a single component, then the invariant vanishes. The problem basically has to do with the crossings involving both components. At the end of this section we will indicate a refinement of this question, involving the idea of "mixed type".

Examples. 1. The simplest examples are those obtained by taking $d \equiv 0$. Then $\chi_{d,f}$ takes the value $f(n)$ on any link with linking number n. This is a degenerate case of the invariants we constructed, since it is of Type 0 on $\mathscr{S}$. It is of Type 1 on $\bar{\mathscr{S}}$ unless f is constant, in which case $\chi_{d,f}$ has Type 0. For example, if $f(n) = n$, then $\chi_{d,f}$ is the linking number.

2. The next simplest examples are those obtained by taking d to be a nonzero constant function. Theorem 5.1 below shows that in this case, $\chi_{d,f}$ is type 3 if and only if f is a polynomial of degree less than 4, and that $\chi_{d,f}$ does not have finite type if f is not a polynomial. The invariant λ corresponds to $d(n) = 1$ and, according to Corollary 3.2, $f(n) = 0$ for all n.
3. A complicated but important example is the Casson–Walker invariant, or more precisely the invariant which assigns to a link with nonzero linking number the Casson–Walker invariant of 0-surgery on each component. This corresponds, according to Theorem A and Corollary 3.2, to $d(n) = -2/n^2$ and $f(n) = -(n^2-1)/(6n)$ if $n \neq 0$. Corollary 5.3 below states that this invariant is not of finite type.

Before we state the main theorem of this section, we should make one point. Since f is defined in terms of $\{T(n)\}$, the choice of $\{T(n)\}$ as base points is now essential. The reader can check that our arguments apply for any family $\{K(n)\}$ which has the following two properties:

1. The linking number of $K(n)$ is n.
2. For each k there exists a singular link L with k double points so that every embedded link obtained by resolving the double points of L is one of the $K(n)$.

Of course $\{T(n)\}$ and $\{T'(n)\}$ are the two most obvious choices of such a family. We chose $T(n)$ because it is simpler in some ways, for example smoothing one of the crossings in $T(n)$ yields the trivial knot.

5.1. THEOREM. *Let $d, f: \mathbf{Z} \to R$ be two functions defining the invariant $\chi_{d,f}$ for 2 component links. View $\chi_{d,f}$ as an invariant on the space $\bar{\mathcal{S}}$ of all singular two-component links. Then*

1. *If $\chi_{d,f}$ is a type n invariant, then d is a polynomial of degree less than $n-1$ and f is a polynomial of degree less that $n+1$.*
2. *If d is nonzero, then $\chi_{d,f}$ either has type greater than 2 or else is not of finite type.*
3. *If d is a constant, nonzero function and $n \geqslant 3$, then $\chi_{d,f}$ is a type n Vassiliev invariant if and only if f is a polynomial of degree less than n.*

Before proving this, we note two consequences.

Proof of Theorem D. The invariant λ corresponds to $d(n) = 1$ and $f(n) = 0$, and so Theorem 5.1(3) implies that the extension to λ to $\bar{\mathcal{S}}$ has Type 3. 5.1(2) says that λ does not have Type 2. ■

5.2. COROLLARY. *Any invariant χ of two-component links whose value on L is the Casson–Walker invariant of 0-surgery on L when the linking number of L is nonzero is not a finite type invariant.*

Proof. The invariant has $d(n) = 2/n^2$ for $n \neq 0$. This is not a polynomial. ■

Before we prove Theorem 5.1, we introduce some convenient notation. The components of our link will be denoted by K and L, and a "KK crossing" will refer to a double point of the component K with itself, and a "KL crossing" will denote a double point involving both components. Moreover, the statement "$\chi_{d,f}(KK, n(KL)) = 0$" will mean that for a particular choice of d and f, the invariant $\chi_{d,f}$ vanishes on every singular link which has one KK double point and n KL double points.

Proof of Theorem 5.1. To begin with, give the set of functions from $\mathbf{Z}$ to R a $\mathbf{Z}[t, t^{-1}]$ module structure by the rule

$$(t^k \cdot f)(n) = f(n-k).$$

Then a simple computation shows that f is a polynomial of degree less than n if and only if $(1-t)^n \cdot f = 0$.

Now suppose that d, f define a type n invariant $\chi_{d,f}$. Then $\chi_{d,f}$ vanishes on the singular links of Fig. 6(a). This link has $n-1$ KL double points and one KK double point. In the figure the box with the dot in it represents twisted strands with alternating double points. The case of $n = -2$ is illustrated in Fig. 6(b).

By resolving all the double points except the KK double point and using the crossing change formula $d(n)\, k(n-k)$ one concludes that

$$0 = \sum_{j=0}^{n-1} (-1)^{n-1-j} \binom{n-1}{j} d(j-k+1)(j-k).$$

(Notice that for each of the 2^{n-1} resolutions, the linking number of L with one lobe of K is equal to 1.)

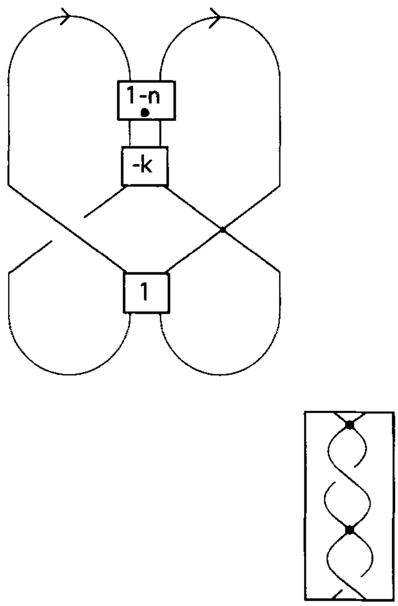

Fig. 6.

This formula can be written in terms of the $\mathbf{Z}[t, t^{-1}]$ action as

$$t^{n-1}(1-t)^{n-1}[(t^{-1}d)\cdot \mathrm{Id}] = 0.$$

Hence, the function $k \mapsto d(k+1)\cdot k$ is polynomial of degree less than $n-1$, and so d is a polynomial of degree less than $n-2$.

We can similarly isolate f by considering the n crossing link of Fig. 7.

Resolving all the double points (which are all KL double points) gives links $T(j)$ for various j. A computation similar to the previous one shows that $(1-t)^n f = 0$, so that f is a polynomial of degree less than n. Thus the first assertion of Theorem 5.1 is proved.

We now prove the second assertion. Let d, f be given defining $\chi_{d,f}$. So $f(n) = \chi_{d,f}(T(n))$. Define a function $g: \mathbf{Z} \to R$ by

$$g(n) = \chi_{d,f}(T'(n))$$

where $\{T'(n)\}$ is the family of links obtained by reversing the orientation of one component.

We computed in Section 3 that

$$f(n) - g(n) = \chi_{d,f}(T(n)) - \chi_{d,f}(T'(n)) = -d(n)\cdot \frac{n(n^2-1)}{6}.$$

Suppose that $\chi_{d,f}$ were type 3. By the first assertion of the theorem $d(n)$ is a polynomial. Consider again Fig. 7, setting $n = 3$. We showed above that $(1-t)^3 f = 0$. But reversing the orientation of one component gives a singular link all of whose resolutions are $T'(j)$ for various j, hence the same argument shows that $(1-t)^3 g = 0$. This implies that $(1-t)^3(f-g) = 0$, and so $f - g$ is a polynomial of degree less than 3. This is a contradiction since $f - g$ is a polynomial of degree at least 3.

The last assertion of the theorem is more difficult, since it involves all links, not just the $T(n)$. The reduction to understanding the $T(n)$ will be achieved by means of the following lemma.

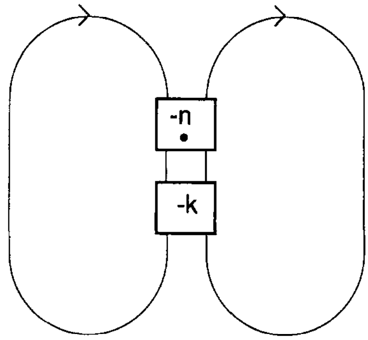

Fig. 7.

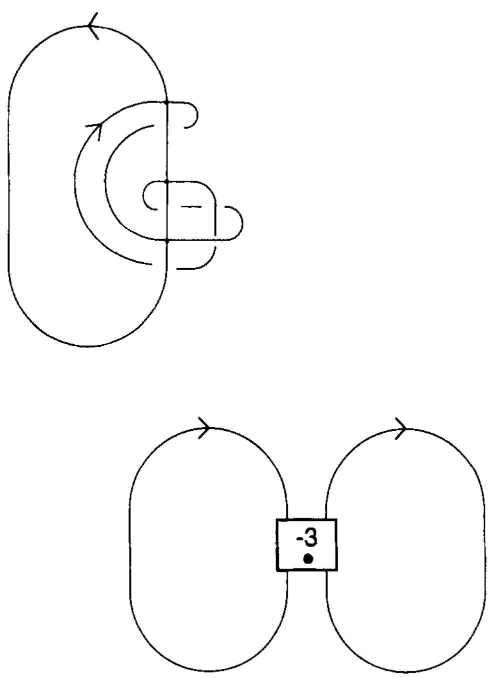

Fig. 8.

5.3. LEMMA. *Suppose that d is the constant function* $d(k) = 1$. *Then for any* f:
1. $\chi_{d,f}(KK, KL)$ *equals the linking number of* L *with the lobe of the* K *which misses* L.
2. $\chi_{d,f}(KK, KL, KL)$ *equals zero or* 1 *according to whether or not one lobe of* K *is disjoint from* L.
3. $\chi_{d,f}(KK, n(KL)) = 0$ *for all* $n > 2$.
4. *The value of* $\chi_{d,f}$ *on the two singular links pictured in* Fig. 8 *differ by* -1.
5. *Given any singular link* (K, L) *with* 4 *or more* KL *double points, let* (K', L) *denote the link obtained from* (K, L) *by the local move illustrated in* Fig. 9. *Then the values of* $\chi_{d,f}$ *on* (K, L) *and* (K', L) *are equal.*

Proof. Parts 1–3 are routine and each implies the next.

Part 4 is slightly more complicated. The value of χ on each can be expressed as the difference of χ on two links, each with two double points. A series of type KK and LL

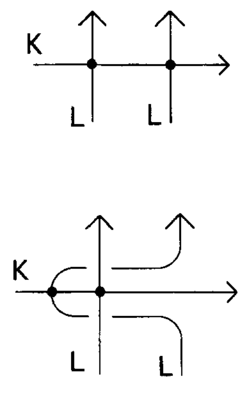

Fig. 9.

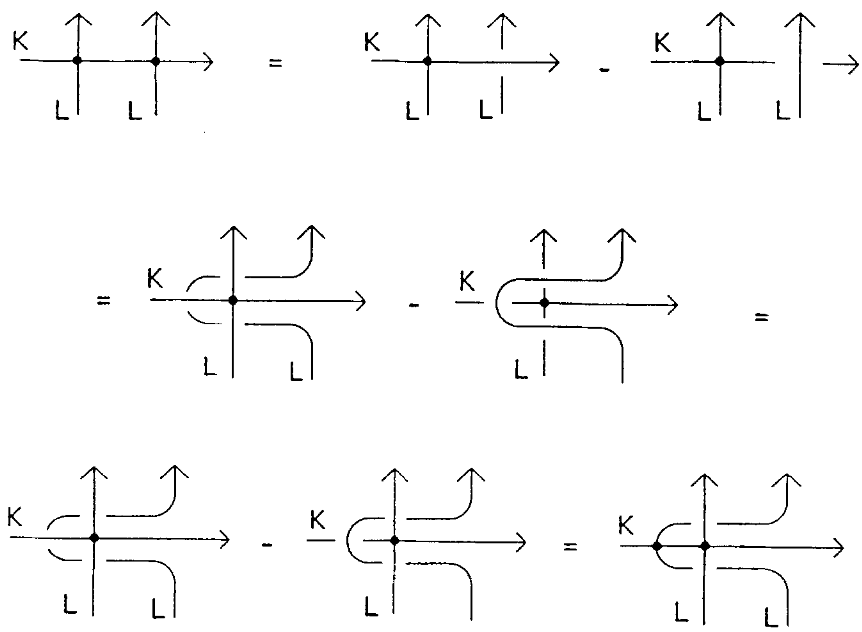

Fig. 10.

crossing changes converts one of the pairs into the other. The effect of these can be computed using part 2. Details are again left to the reader.

For the last assertion, Fig. 10 illustrates the proof schematically. In the figure, each small diagram is taken to mean the value of χ on a link with the given local picture. Each equality is obvious, except perhaps the third; in this case the altered links differ by a LL crossing change, so that by part 3, there is no change in the value of χ. ∎

We can now finish the proof of Theorem 5.1. Suppose that d is constant and nonzero, by rescaling assume that $d = 1$.

Let $n > 2$, and let (K, L) be any link with n double points, all of KL type. Then L can be made into an unknotted curve by introducing only LL crossings, which do not affect the value of $\chi_{d,f}$ by the 3rd assertion of the lemma. Using the 4th assertion of the lemma, we may assume that the double points are ordered in any way we wish on K and L without changing the value of $\chi_{d,f}$. In particular, we may assume that ordering the double points clockwise on K orders them counterclockwise on L. Let k denote the linking number of the embedded link obtained by resolving all double points negatively. Then by introducing only KK double points, we can change the link to the link pictured in Fig. 7 without changing the value of $\chi_{d,f}$, using part 3 of the lemma again.

But the link in Fig. 7 has $\chi_{d,f}$ equal to

$$\sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} f(j+k) = [(1-t)^n \cdot f](-k).$$

Thus $\chi_{d,f}$ has type n if and only if $(1-t)^n \cdot f = 0$, and hence if and only if f is a polynomial of degree less than n. This finishes the proof of the theorem. ■

From the proof of Theorem 5.1 and in particular Lemma 5.3 one gleans that the notion of "type" can be refined for link invariants of 2 component links to be a subset of $\{-1, 0, 1, 2, \dots\}^3$. This is done by saying that an invariant has *Type* (p, q, r) provided it vanishes on any link that has $p+1$ KK crossings, $q+1$ LL crossings, and $r+1$ KL crossings. For example, our invariant λ has Type $(1, -1, -1)$ and also has Type $(0, 0, -1)$ and $(-1, 1, -1)$. Lemma 5.3 says that this also has Type $(0, -1, 3)$ and $(-1, 0, 3)$. Since an invariant of Type (p, q, r) also has Type $(p+a, q+b, r+c)$ for $a, b, c \geqslant 0$, the type is the intersection of the integer lattice with a union of translates of the first octant in $\mathbf{R}^3$. Thus one can pose a more refined problem: describe the type of $\chi_{d,f}$ in this sense.

6. COMPUTING CASSON–WALKER INVARIANTS, PERIODICITY, AND THE CONWAY POLYNOMIAL

A. Computing Casson–Walker invariants

One application of our results is Theorem E, which gives a simple combinatorial method of computing the Casson–Walker invariant for manifolds obtained as 0-surgery on a two component link. The proof is a generalization of the computation of the example in Section 2.

Proof of Theorem E. The difference $\lambda(L) - \lambda(L')$ is equal to $(n^2/2)(\mu(L) - \mu(L'))$ by definition. However, M_L is diffeomorphic to $M_{L'}$ via an orientation reversing diffeomorphism. Hence $\mu(L') = -\mu(L)$. The formula follows. ■

Example. The difference $\lambda(T(n)) - \lambda(T'(n))$ is computed in Section 2 to be $\frac{1}{6}(n - n^3)$. Hence, by the last theorem, the Casson–Walker invariant of 0-surgery on the $(2, n)$-torus link is $(1 - n^2)/6n$. This computation was already used (as Corollary 3.2). The example at the end of Section 3 gives another illustration of this approach to calculating the Casson–Walker invariant.

B. Amphicheirality

A link L is called *amphicheiral* if it is isotopic to its mirror image, L^m, ignoring the orientation and ordering of the components.

Proof of Theorem F. M_L and M_{L^m} are homeomorphic manifolds via an orientation reversing homeomorphism. Hence, $\mu(M_L) = -\mu(M_{L^m})$. However, if L is amphicheiral, then M_L is orientation-preserving homeomorphic to M_{L^m}. Therefore, $\mu(M_L) = 0$. ∎

6.1. COROLLARY. *If L is amphicheiral, then the linking number of L cannot equal 2 modulo 4.*

Proof. Notice that $\lambda(L)$ is an integer. But if $n \equiv 2 \pmod 4$, then $\frac{1}{12}(n^3 - n)$ is not an integer. ∎

Example. For any odd n there is an amphicheiral link with linking number n. For example, when $n = 5$ the braid $\sigma_1\sigma_2\sigma_4^{-1}\sigma_3^{-1}$ together with its axis is easily seen to be amphicheiral.

C. Periodicity

A knot K is *periodic of period d with axis U* if there is an orientation preserving order d diffeomorphism of S^3 with fixed point set U that leaves K invariant. By the solution of the Smith conjecture, U is necessarily unknotted in this case. Here we show that λ provides an obstruction for a knot to be periodic.

6.2. THEOREM. *Let K and K' be two periodic knots of period k with axis U and linking number with the axis n. The difference $\lambda(K, U) - \lambda(K', U)$ is divisible by k or $2k$, depending on whether n is even or odd, respectively.*

Proof. Consider the quotient links, (J, U) and (J', U). A series of crossing changes applied only to J changes (J, U) into (J', U). Each crossing change downstairs determines k changes in K upstairs, each with the same value of $\kappa(n - \kappa)$. Doing all the lifted crossing changes to K yields K', and the desired result follows. ∎

Application. For the link L of linking number 1 in Fig. 11, the difference $\lambda(L) - \lambda(T(1)) = 2$. Since $T(1)$ is period k for all k, L can have no period k.

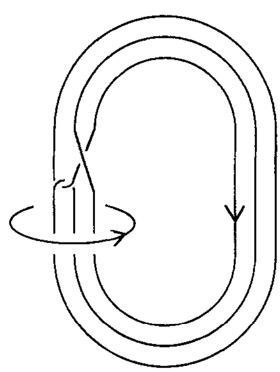

Fig. 11.

D. Relation to the Conway polynomial

6.3. THEOREM. *The invariant λ on $\mathcal{S}_n$ is congruent modulo n to the 3rd coefficient c_3 of the Alexander–Conway polynomial. Moreover, $\lambda(T(n)) = 0 = c_3(T(n))$. However, given any multiple cn of n there exists a link L such that $\lambda(L) - c_3(L) = cn$.*

Proof. Corollary 3.2 shows that $\lambda(T(n)) = 0$. Smoothing a crossing between components of $T(n)$ gives the trivial knot. Using the crossing change formula $\nabla(L_+) - \nabla(L_-) = -z\nabla(L_0)$ for the Conway polynomial (where L_0 denotes the "smoothed" link) one concludes that $c_3(T(n)) = c_3(T(n-1))$ and by induction $c_3(T(n)) = 0$. This, together with Corollary 3.2, proves the second assertion of the theorem.

Thus to prove the first assertion Theorem 6.2, we need only show that c_3 satisfies the crossing change formula $\kappa(n-\kappa)(\operatorname{mod} n)$ for a linking number n link, with κ as in the introduction.

The crossing change formula for the Conway polynomial implies that

$$c_3(L_+) - c_3(L_-) = c_2(L_0).$$

The expression on the right is the second coefficient of the Conway polynomial of the three-component link L_0. But a routine computation shows that for a three-component link with pairwise linking numbers a, b, and c, $c_2 = ab + ac + bc$. In the present case, we can take $a = \kappa$, $b = n - \kappa$ and then c is the linking number of the two lobes. Thus

$$ab + ac + bc = \kappa(n-\kappa) + \kappa c + (n-\kappa)c \equiv \kappa(n-\kappa)(\operatorname{mod} n), \qquad (*)$$

proving the first assertion.

The last assertion is obtained by applying (*). Starting with $T(n)$, one can easily change a crossing so that the two lobes of the intermediate singular link have linking number c but with $\kappa = 0$. This does not affect the value of λ, and so by (*) the resulting link L satisfies $c_3(L) - \lambda(L) = cn$. ■

Note, the third coefficient of the Conway polynomial is Type 3 on $\bar{\mathcal{S}}$ and is Type 2 on $\mathcal{S}$.

Addenda—Kanenobu, Miyazawa, and Tani have classified type 3 link invariants for (not necessarily disjoint) links in S^3. As a consequence they observe that $\lambda(L) = c_3(L) - c_1(L)(c_2(L_1) + c_2(L_2))$. (A simple derivation of this formula follows by checking that this function of Conway coefficients of links satisfies our crossing change formula and that it vanishes on $T(n)$, as required by Corollary 3.2.) This offers an alternative proof of the existence of a nontrivial type 1 invariant on the space of disjoint links. It also completes the analysis begun in Theorem 6.3 of the present paper. Eiji Ogasa has noticed that it implies that $2\lambda(L) = \beta^*(L) + \operatorname{lk}(L_1, L_2)/2 \bmod 4$, where $\beta^*(L)$ is the Z_4 valued extension of the Sato–Levine invariant, defined for links of even linking number by M. Saito ("On the unoriented Sato–Levine invariant," *J. Knot Theory Ramifications 2*, No. 3, pp. 335–358, 1993).

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